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On the Construction of Analytic and Crossing Symmetric Partial Waves[1]

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Abstract

The problem of construction of analytic and crossing symmetric partial waves leads to a generalized Riesz problem. In some cases nontrivial solutions can be constructed from the solutions of a homogeneous Fredholm integral equation. It is suggested that the unitary partial waves satisfy a similar equation.

1. *Introduction*

The general properties of the pion-pion scattering amplitude, analyticity, crossing symmetry, and unitarity take a remarkably simple form when they are applied to partial waves. In spite of this, nobody has till now constructed functions that satisfy them exactly.

In contradistinction to the previous approaches we suggest constructing partial waves since on them the above general requirements can be turned into true equations.

In our approach the problem will have two steps: In the first one the linear part of the constraints will be solved, i.e., the problem of the description of the class of functions that are analytic in a given domain of the complex s plane and satisfy the Balachandran-Nuyts-Roskies (BNR) crossing relations. The second step will consist in finding inside this class those partial waves that are unitary.

It will be shown that the linear part of the problem is equivalent to the generalization of a problem first raised by Riesz (Adamyan *et al.*, 1968). We have succeeded in solving this problem only in some special cases. In the others we suggest a way that may lead to the solution. The full solution requires the solving of a separation problem for the spectrum of a Hankel operator.

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It happens that the analyticity and the crossing symmetry together give rise to a Hankel operator that has remarkable properties. With the Schmidt pairs $\xi, \eta, (\xi, \eta \in l_2)$ of this operator we can construct nontrivial functions that are exactly crossing symmetric and analytic. In this case the partial waves can be constructed from the solutions of a homogeneous Fredholm integral equation.

In the first part of the paper we show the equivalence of the linear part of the problem with the Riesz problem and give the description of the associated Hankel operator, and in the second part we construct solutions to our problem.

2. *Statement of the Problem*

The key point in the following will be the change of the partial wave crossing relations into true equations. The BNR relations (Balachandran and Nuyts, 1968; Roskies, 1970) are necessary and sufficient conditions for crossing, and this is the reason we will consider them as true equations, in contradistinction to the usual view, which takes them as sum rules.

Their general form is (Roskies, 1970)

$$\int_0^4 (4-s)F_n(s)\,ds = 0, \qquad n = 1, 2, \ldots$$

where

$$F_n(s) = \sum_{l,I} C^I_{n,l}(s) f_l^I(s)$$

$f_l^I(s)$ being the partial waves and $C^I_{n,l}(s)$ given functions. So we will consider only one relation, written in the form

$$\int_0^4 (4-s)f(s)ds = 0 \tag{2.1}$$

where by $f(s)$ we will understand a "partial wave."

The partial waves are analytic functions in a domain D of the complex s plane that includes the interval $(0, 4)$, the exact shape of it being given by Martin (Martin, 1969).

Let D' be another domain such that $D' \subset D$, which can be D minus a finite region around the left-hand cut. Let $z(u)$, $u = (s-2)/2$ be the conformal mapping of D' onto the unit disk $|z| < 1$ such that $z(0) = 0$.

We define the function

$$g(z) \equiv S(s) = 1 + \sqrt{(4-s)/s}\,f(s) \tag{2.2}$$

and remark that $g \in H_1$, where H_1 is the Hardy space of functions $f(z)$ regular in the unit disk whose norms are given by

$$\|f\|_1 = \sup_{r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\,d\theta\right\}$$

Then the relation (2.1) gives us

$$\int_0^4 \sqrt{s(4-s)}S(s)ds = 2\pi \qquad (2.1')$$

In fact we must apply the transformation (2.2) to the true partial waves, but then the right-hand side of (2.1′) is modified by a trivial factor. Let

$$g(z) = \sum_{n=1}^{\infty} a_n z^{n-1}$$

be the Taylor expansion of the function $g(z)$. By putting it into (2.1′) one gets

$$\sum_{n=1}^{\infty} a_n \gamma_n = 1$$

where

$$\gamma_n = \frac{1}{2\pi} \int_0^4 \sqrt{s(4-s)} z^{n-1}(s)\, ds, \qquad n = 1, 2, \ldots$$

This is the form of the crossing symmetry we have looked for.

Now we can state the problem:

> *The Riesz Generalized Problem.* Given a positive number $\rho > 0$ and an infinite sequence $\{\gamma_n\}_1^\infty$ of complex numbers it is required to find among the functions
>
> $$f(z) = \sum_{n=1}^{\infty} a_n z^{n-1}$$
>
> regular in the disk $|z| < 1$ those for which
>
> $$\sum_{n=1}^{\infty} a_n \gamma_n = 1 \qquad (2.3a)$$
>
> $$\|f\|_1 \leqslant \rho \qquad (2.3b)$$

This problem is the generalization of a problem by Riesz originally stated for a finite sequence $\{\gamma_n\}_1^N$ and the minimum norm in the H_1 space (Adamyan *et al.*, 1968).

If we want to extend this treatment to the pion–nucleon scattering we must generalize the above problem since the partial waves of the πN reactions can have poles on the first sheet.

If $f \in L_1(0, 2\pi)$, we will define the Fourier coefficients by

$$c_k(f) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta)\zeta^k \, d\theta, \qquad \zeta = e^{i\theta}, \quad k = 0, \pm 1, \ldots \qquad (2.4)$$

and the generalization of the above problem would be as follows:

Given a positive number $\rho > 0$, an integer $k \geqslant 0$ and an infinite sequence $\{\gamma_n\}_1^\infty$ of complex numbers, it is required to find those functions $f(\zeta) \in L_1(0, 2\pi)$ for which

$$\sum_{n=0}^{\infty} c_{-n}(f)\gamma_{n+1} = 1 \tag{2.3a'}$$

$$\|f\|_1 \leqslant \rho \tag{2.3b'}$$

$$\sum_{l=1}^{\infty} c_l(f)\zeta^{-l} \tag{2.3c}$$

is a rational function with at most k poles inside the unit disk.

Since the pion-pion partial waves are real analytic functions, the domain D, by the reflection principle, will be a symmetric domain, so we can take D' also symmetric and then we can choose the phase of the conformal mapping such that the numbers γ_n will be real numbers.

In the following we will suppose that γ_n are real numbers.

With the sequence $\{\gamma_n\}_1^\infty$ we will associate the Hankel matrix $\Gamma = (\gamma_{j+k-1})_1^\infty$. This operator, generated by both crossing and analyticity, has remarkable properties given by Theorem 1:

Theorem 1. The Hankel operator $\Gamma = (\gamma_{j+k-1})_1^\infty$ is a trace class operator. Its spectrum is non-negative and the eigenvalues $\lambda \neq 0$ are simple.

Proof. Let $z(u)$, $u = (s-2)/2$ be the conformal mapping of D' onto the unit disk $|z| < 1$, with $z(0) = 0$. We choose the phase such that the interval $(0, 4)$ will go into $(-1, 1)$.

By the Schwarz lemma

$$|z(u)| \leqslant |u| = |(s-2)/2| \qquad \text{for } |u| < 1$$

From the relation of definition for γ_n we have

$$2\pi|\gamma_n| = \left|\int_0^4 \sqrt{s(4-s)}\, z^{n-1}(s)\, ds\right| = \left|4\int_{-1}^{1} \sqrt{1-u^2}\, z^{n-1}(u)\, du\right|$$

$$\leqslant 4\int_{-1}^{1} \sqrt{1-u^2}\, |z^{n-1}(u)|\, du \leqslant 4\int_{-1}^{1} \sqrt{1-u^2}\, |u|^{n-1} du = 4B\left(\frac{n}{2}, \frac{3}{2}\right)$$

and this implies

$$2\pi \sum_{n=1}^{\infty} |\gamma_n| = 4\left[B\left(\frac{1}{2}, \frac{1}{2}\right) + B\left(1, \frac{1}{2}\right)\right] < \infty \tag{2.5}$$

which shows that Γ is a trace class operator. The condition (2.5) is a sufficient condition for compacticity of Γ in the l_1 space. But then by a known lemma

(Adamyan *et al.*, 1971) the operator Γ has the same eigenvectors in all l_p spaces. Hence if $\xi = (\xi_j)_1^\infty$ is an eigenvector of Γ, the function

$$\xi(z) = \sum_{j=1}^{\infty} \xi_j z^{j-1}$$

is an analytic function in $|z| < 1$ and continuous on $|z| = 1$.

Let $\xi = (\xi_j)_1^\infty$ be a non-null eigenvector ($\|\xi\|_2 \neq 0$) of Γ that satisfies the eigenvalue equations

$$\sum_{m=1}^{\infty} \gamma_{n+m-1}\, \xi_m = \lambda \xi_n, \qquad n = 1, 2, \ldots$$

or in an equivalent form

$$\sum_{m=1}^{\infty} \left[\int_{-1}^{1} z^{n+m-2}(u)\sqrt{1-u^2}\, du \right] \xi m = 2\pi\lambda\xi n, \qquad n = 1, 2, \ldots \quad (2.6)$$

Since

$$\xi(z) = \sum_{j=1}^{\infty} \xi_j z^{j-1}$$

is an analytic function continuous on $|z| = 1$ we can change the sum and the integral in (2.6) and obtain

$$\int_{-1}^{1} \sqrt{1-u^2}\, z^{n-1}(u)\xi(z)\, du = 2\pi\lambda\xi_n, \qquad n = 1, 2, \ldots$$

Multiplying by ξ_n and by taking the sum one gets

$$\lambda = \frac{1}{2\pi} \frac{\int_{-1}^{1} \sqrt{1-u^2}\, \xi^2(z)\, du}{\|\xi\|_2^2} \geqslant 0$$

i.e., the spectrum is non-negative.

Let $n \geqslant 1$ be the multiplicity of the eigenvalue $\lambda > 0$. Let $u_1, u_2, \ldots, u_n$ be the corresponding eigenvectors. Then there exist some constants $c_1, c_2, \ldots, c_n$ such that the function $u = \Sigma c_i u_i$, which is an eigenvector too, has a zero of order $n - 1$ at the point $z = 1$. If n_1 is the multiplicity of λ and n_2 that of $-\lambda$ in the point spectrum of Γ, a theorem by Clark (Clark, 1968) states that the number of zeros of $u(z)$ at z_0, where $|z_0| = 1$, is $p \leqslant 2\min(n_1, n_2)$. Since $n_2 = 0$ we obtain $n \leqslant 1$. Q.E.D.

3. *Construction of Solutions*

Let $\Gamma = (\gamma_{j+k-1})_1^\infty$ be a compact Hankel operator. The s numbers of a (non-Hermitian) operator are the eigenvalues of the non-negative operator $(\Gamma^+\Gamma)^{1/2}$. In our case $\Gamma^+ = \bar{\Gamma}$, the bar meaning the complex conjugation.

Since the operator Γ is compact the multiplicity of eigenvalues is finite. Let $\|\Gamma\| = s_0 \geqslant s_1 \geqslant \cdots \geqslant s_k \geqslant \cdots$ be the enumeration of the spectrum of $(\bar{\Gamma}\Gamma)^{1/2}$, the eigenvalues being included with their multiplicities. Then for $s = s_k(\Gamma)$ there exists a p-dimensional linear manifold of Schmidt pairs (ξ, η), $\xi, \eta \in l_2$ such

$$\Gamma\xi = s\eta$$
$$\bar{\Gamma}\eta = s\xi$$

where p is the multiplicity of the s number s_k.

If $\xi = (\xi_j)_1^\infty$ and $\eta = (\eta_j)_1^\infty$, $\xi, \eta \in l_2$ we will define the functions

$$\xi_+(\zeta) = \sum_1^\infty \xi_j \zeta^{j-1}, \qquad \eta_-(\zeta) = \sum_1^\infty \eta_j \zeta^{-j}$$

If $s = s_k$ is an eigenvalue of the non-negative operator $(\bar{\Gamma}\Gamma)^{1/2}$ with multiplicity p the whole set of Schmidt pairs (ξ, η) corresponding to this s is given by (Adamyan *et al.*, 1971)

$$\begin{aligned} \xi_+(\zeta) &= P(\zeta)a(\zeta)\varphi_e(\zeta) \\ \eta_-(\zeta) &= \zeta^{-p}P(\zeta)\overline{a(\zeta)}\,\overline{\varphi_e(\zeta)} \end{aligned} \tag{3.1}$$

where $P(\zeta)$ is an arbitrary polynomial of degree less than p, $a(\zeta)$ is an inner function that has exactly k zeros and $\varphi_e(\zeta)$ is an outer function.

With the sequence $(\gamma_j)_1^\infty$ we will associate the set $M(\Gamma)$ of all the functions $f \in L_\infty$ for which $c_k(f) = \gamma_k$, $k = 1, 2, \ldots$ where $c_k(f)$ are the Fourier coefficients of f defined by the relation (2.4), and the continuous functional on H_1

$$\phi_\Gamma(g) = \lim_{n\to\infty} \sum_{j=0}^{n} \left(1 - \frac{j}{n}\right) \gamma_{j+1} c_{-j}(g), \qquad g \in H_1$$

It is easily shown (Adamyan *et al.*, 1968) that for any $\Gamma(\|\Gamma\| < \infty)$ and any $f \in M(\Gamma)$ we have

$$\phi_\Gamma(g) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta)g(\zeta)\zeta \, d\theta, \qquad \zeta = e^{i\theta}, \quad g \in H_1$$

and for any $\xi, \eta \in l_2$

$$(\Gamma\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta)\xi_+(\zeta)\overline{\eta_-(\zeta)}\, d\theta = \phi_\Gamma(\xi_+\overline{\eta_-\zeta})$$

If $\xi, \eta \in l_2$ are such that $(\Gamma\xi, \eta) \neq 0$ the function

$$g(\zeta) = \frac{\xi_+(\zeta)\overline{\eta_-(\zeta)}\bar{\zeta}}{(\Gamma\xi, \eta)} = \frac{\xi_+(\zeta)\overline{\eta_+(\bar{\zeta})}}{(\Gamma\xi, \eta)} \tag{3.2}$$

will satisfy the relation (2.3a), hence it will be a crossing symmetric function. This shows how many solutions have the crossing relations.

Now we can construct all the solutions to our problem when the condition (2.3b) has the form

$$\|f\| = \rho = s_k^{-1}, \qquad k = 0, 1, 2, \ldots \tag{2.3b$''$}$$

Indeed for $\rho = s_k^{-1}$ we have the Schmidt pairs in the form (3.1). We will suppose they are normalized $\|\xi\| = \|\eta\| = 1$. Then the function

$$f(z) = \frac{\xi_+(z)\overline{\eta_+(\bar{z})}}{(\Gamma\xi, \eta)} = s_k^{-1}\,\xi_+(z)\,\overline{\eta_+(\bar{z})}$$

satisfies the relation (2.3a) and

$$\|f\|_1 = s_k^{-1}\,\frac{1}{2\pi}\int_0^{2\pi} |\xi_+(\zeta)\overline{\eta_+}(\bar{\zeta})|\,d\theta = s_k^{-1}\,\frac{1}{2\pi}\int_0^{2\pi} |\xi_+(\zeta)|^2 d\theta = s_k^{-1}$$

i.e., $f(z)$ satisfies also the condition (2.3b$''$).

Since the set of the whole Schmidt pairs is given by (3.1) the functions

$$f(z) = s_k^{-1} z^{p-1} P(z)\overline{P(1/\bar{z})}\,a^2(z)\varphi_e^2(z) \tag{3.3}$$

will give all the solutions of our problem satisfying the conditions (2.3a) and (2.3b$''$).

The multiplicity of every eigenvalue being unity, the form (3.3) simplifies to

$$f(z) = s_k^{-1}\,|A|^2 a^2(z)\varphi_e^2(z) \tag{3.3$'$}$$

where the constant $|A|$ is determined from the normalization condition $\|\xi\| = \|\eta\| = 1$.

The eigenvalues of our (Hermitian) operator Γ being positive and simple the s numbers will be given by the eigenvalues of Γ:

$$\Gamma\xi = s\xi$$

or in another form

$$\sum_{m=1}^{\infty} \gamma_{n+m-1}\,\xi_m = s\xi_n, \qquad n = 1, 2, \ldots \tag{3.4}$$

The eigenvalue equation (3.4) is equivalent to an integral equation.

Let $\varphi(x)$ be the function

$$\varphi(x) = \sum_{j=1}^{\infty} \xi_j x^{j-1}$$

where $\xi = (\xi_j)_1^\infty$ is a solution of (3.4). By taking into account the form of γ_n it is easy to show that the equation (3.4) is equivalent to the following one:

$$\frac{1}{2\pi}\int_{-1}^{1} \sqrt{1-u^2}\,\frac{\varphi(z(u))}{1 - wz(u)}\,du = s\varphi(w)$$

Let $u = h(z)$ be the inverse function of the conformal mapping $z = z(u)$. By a change of variable we obtain

$$\frac{1}{2\pi} \int_{-1}^{1} dx h'(x)\sqrt{1 - h^2(x)} \, \frac{\varphi(x)}{1 - xy} = s\varphi(y) \tag{3.5}$$

which is the integral equation we have looked for. So we have shown the following:

Theorem 2. The functions $f(z) = s_k^{-1}\varphi_k^2(z)$, $k = 0, 1, 2, \ldots$ where s_k and $\varphi_k(z)$ are the eigenvalues and the corresponding (orthonormal) eigenfunctions of the homogeneous Fredholm integral equation (3.5) give the whole solutions of the BNR crossing relations whose norms are exactly s_k^{-1}. For every s_k there exists only one solution.

If $\|f\|_1 < \|\Gamma\|^{-1}$ we have no solution (Adamyan *et al.*, 1968). For $\rho \neq s_k^{-1}$, $k = 0, 1, 2, \ldots$ we have an infinity of solutions. The description of these solutions was not obtained.

In the following we suggest a method that may give the answer.

Let $\psi(z)$ be a bounded function $\psi(z) \in H_\infty$. Let T be the shift operator in the l_2 space. The action of T on $\xi = (\xi_1, \xi_2, \ldots)$ is $T\xi = (0, \xi_1, \ldots)$ and of its adjoints T^+, $T^+\xi = (\xi_2, \ldots)$. A Hankel operator being an operator for which $T^+\Gamma = \Gamma T$, it is evident that the operator $\Gamma' = \Gamma\psi(T)$ is also a Hankel operator. Now if $g(z) \in H_1$ it is easily seen that

$$\phi_{\Gamma\psi(T)}(g(\zeta)) = \phi_\Gamma(\psi(\zeta)g(\zeta)) \tag{3.6}$$

for every bounded function $\psi(z) \in H_\infty$ and any bounded Γ. The last relation gives us the possibility of constructing the solutions of the problem when $\rho \neq s_k^{-1}$ if we know the solution to the following (unsolved) problem:

Let Γ be a compact Hankel operator with the s numbers $\|\Gamma\| = s_0 \geqslant s_1 \geqslant \cdots \geqslant s_{k-1} > s_k \geqslant \cdots \geqslant 0$ and $\rho > 0$ such $s_{k-1} > \rho^{-1} > s_k$. It is required to find all the inner functions $\psi(z) \in H_\infty$ for which the Hankel operator $\Gamma' = \Gamma\psi(T)$ has ρ^{-1} as an s number, and the corresponding (ξ, η) Schmidt pairs.

$$\begin{aligned} \Gamma\psi(T)\xi &= \rho^{-1}\eta \\ \overline{\Gamma\psi(T)}\eta &= \rho^{-1}\xi \end{aligned} \tag{3.7}$$

The theorem contained in the appendix of Clark's paper (Clark, 1968) shows that such functions do exist, and, moreover, it suggests that the inner functions $\psi(z)$ can be simple Blashke factors.

If we know the solution of the above problem the functions

$$g(z) = \frac{\xi_+(z)\overline{\eta_+(\bar{z})}}{(\Gamma\psi(T)\xi, \eta)} = \rho\,\xi_+(z)\overline{\eta_+(\bar{z})}, \qquad \|\xi\| = \|\eta\| = 1$$

will satisfy the relation

$$\phi_{\Gamma\psi(T)}(g) = 1 = \phi_\Gamma(\psi(z)g(z)) = \phi_\Gamma(h(z))$$

i.e., the newly defined functions $h(z) = \psi(z)g(z)$ will be crossing symmetric and $\| h \|_1 = \| \psi(z)g(z) \|_1 = \| g \|_1 = \rho$. Thus the functions $h(z)$ will give the extremal solutions of the generalized Riesz problem when $\rho \neq s_k^{-1}$. By constructing the linear convex hull of them we will obtain the description of the whole solutions of the BNR crossing relations.

If the H_1 norms of the physical partial waves are not given by $s_k^{-1}(\Gamma)$ (for a given domain D'), then there must exist inner functions $\psi(z)$ such that the norms are given by $s'^{-1}(\Gamma')$ [where s' is an s number of the Hankel matrix $\Gamma' = \Gamma\psi(T)$] and among the functions $h(z) = s'^{-1}\psi(z)\varphi^2(z)$, where $\varphi(z)$ is an (orthonormal) solution of the homogeneous Fredholm integral equation:

$$\frac{1}{2\pi}\int_{-1}^{1} dx h'(x)\sqrt{1-h^2(x)}\,\psi(x)\,\frac{\varphi(x)}{1-xy} = s'\varphi(y) \tag{3.8}$$

there must be at least one that satisfies the physical constraints. This is suggested to us by the solution of the problems solved by Adamyan *et al.* (1971). It seems to be nontrivial that the unitary and crossing symmetric partial waves could be obtained from the solutions of a linear integral equation like (3.8).

We consider that the knowledge of the solution to the generalized Riesz problem will be the decisive step toward the construction of physical partial waves that satisfy exactly all the required properties.

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